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A SIMPLE PROOF OF THE RAMSEY SAVINGS EQUATION*

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In [2] Ramsey considers the problem of minimizing the accumulated difference between bliss B and net utility $U(x)-V(a)$, where $U(x(t))$ is the instantaneous utility of consuming $x(t)$ and where $V(a)$ is the instantaneous disutility of working at the rate $a(t)$.[1] Capital $c(t)$ and labor $a(t)$ are used to produce the output flow according to the production function $f(c(t), a(t))$. Output in turn is divided between capital accumulation, $c(t)$ and consumption $x(t)$. Thus the problem is to minimize the integral[2] $\int_0^T (B-U(x)+V(a))\,dt$ subject to $\dot{c}+x=f(c, a)$. In other words, we are to minimize the integral:

$$\int_0^T (B-U(f(c, a)-\dot{c})+V(a))\,dt.$$

Setting $a=\dot{y}$, the problem becomes:

$$\text{minimize} \int_0^T L(Z, \dot{Z})\,dt$$

where $Z=(Z_1, Z_2)=(c, y)$.

Since L does not depend explicitly on t, the Euler equations have the form (see Akhiezer [1]):

$$\frac{d}{dt}L_{\dot{Z}}=L_Z, \tag{1}$$

$$L-\dot{Z}_1L_{\dot{Z}_1}-\dot{Z}_2L_{\dot{Z}_2}=0. \tag{2}$$

Writing these in terms of our problem we have:

$$\frac{d}{dt}U'=U'f_c, \tag{3.1}$$

$$\frac{d}{dt}(U'f_a-V')=0, \tag{3.2}$$

$$B-U+V-\dot{c}U'-a(U'f_a-V')=0. \tag{3.3}$$

Clearly,

$$U'f_a-V'=0. \tag{3.4}$$

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[1] The purpose of this note is purely pedantic since it reveals no new "truths." It merely makes it easier to understand what is already a widely held belief.

[2] Ramsey considers the case where $T=\infty$. By taking the limit, and arguing around a bit, we can show that the Euler equations in our simple problem will still hold. That, however, will make the proof far from simple. It should also be noted that we don't discuss the existence problem, thus embracing the risk of an empty intersection between the set of arcs for which a solution exists and the set of arcs for which Euler's equations are necessary conditions for an extremum.

Hence, (3.3) becomes

$$B - U + V - \dot{c}U' = 0. \tag{3.5}$$

Consequently,

$$\dot{c} = \frac{B - U + V}{U'}. \tag{4}$$

Equation (4) is Ramsey's savings equation and has the usual interpretation that the "optional" policy is to save more if the net satisfaction is below bliss level and to dissave if that level is surpassed. Equations (3.1) and (3.4) express the usual market conditions for capital and labor.

REFERENCES

[1] N. I. AKHEIZER, *The Calculus of Variations*, Blaisdell, New York, 1962.
[2] F. RAMSEY, *A mathematical theory of savings*, Economic J., 38 (1928), pp. 543–559.